Topic: Fourier series examples
Course: B.Sc/ Physics

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magnitude equal $(-d\phi/ds)_{max}$ ... and its direction

~~increase~~ change of ϕ at that point and its direction is perpendicular to the surface ϕ = constant at that point.

Q. Prove that $x^2 = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \dfrac{\cos nx}{n^2}$, $-\pi < x < \pi$.

(i) $\sum \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$; (ii) $\sum \dfrac{1}{(2n-1)^2} = \dfrac{\pi^2}{8}$. (iii) $\sum \dfrac{1}{n^4} = \dfrac{\pi^4}{90}$

Sol: $\because f(x) = x^2$ is an even fn. $\therefore b_n \to$ are absent

$\therefore a_0 = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{2}{\pi}\int_0^{\pi} x^2\,dx = \dfrac{2}{\pi} \times \dfrac{\pi^3}{3} = \dfrac{2\pi^2}{3}$.

$a_k = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \dfrac{2}{\pi}\int_0^{\pi} x^2 \cos kx\,dx$

$= \dfrac{2}{\pi}\left[\dfrac{x^2}{k}\sin kx + \dfrac{2x}{k^2}\cos kx - \dfrac{2}{k^3}\sin kx\right]_0^{\pi} = \dfrac{2}{\pi} \times \dfrac{2\pi}{k^2}\cos k\pi = \dfrac{4}{k^2}\cos k\pi$

Now $f(x) = x^2 = \dfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx = \dfrac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \dfrac{(-1)^k \cos kx}{k^2}$ — (1)

Putting $x = \pi$: $f(\pi) = \pi^2 = \dfrac{\pi^2}{3} + 4\sum \dfrac{1}{k^2}$ or, $\dfrac{2\pi^2}{3 \times 4} = \sum \dfrac{1}{k^2}$

or, $\boxed{\sum \dfrac{1}{n^2} = \dfrac{\pi^2}{6}}$

(ii) Putting $x = 0$ in (1)

$0 = \dfrac{\pi^2}{3} + 4\sum \dfrac{(-1)^k}{k^2} = \dfrac{\pi^2}{3} + 4\left[-1 + \dfrac{1}{2^2} - \dfrac{1}{3^2} + \dfrac{1}{4^2} - \dots\right]$

or $\dfrac{\pi^2}{3} = 4\left[1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \dots\right]$

or, $\dfrac{\pi^2}{12} = \left(1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \dots\right)$ — (1)

& $\dfrac{\pi^2}{6} = \sum \dfrac{1}{n^2} = \left(1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \dots\right)$ (2)

adding (1) & (2) $\dfrac{\pi^2}{6} + \dfrac{\pi^2}{12} = 2\left[1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \dots\right] = 2\left[\dfrac{1}{(2n-1)^2}\right]$

$\therefore \boxed{\sum \dfrac{1}{(2n-1)^2} = \dfrac{\pi^2}{8}}$

(iii) $\sum \dfrac{1}{n^4} = \left(1 + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \dfrac{1}{4^4} + \dfrac{1}{5^4} + \dfrac{1}{7^4}\right) + \dots = \left(1 + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \dots\right) + \left(\dfrac{1}{2^4} + \dfrac{1}{4^4} + \dots\right)$

$= \sum_{n=odd} \dfrac{1}{n^4} + \dfrac{1}{16}\sum_{n=even} \dfrac{1}{n^4}$

$$f(x) = x^2 = \frac{\pi^2}{3} + 4\sum(-1)^n \frac{\cos nx}{n^2}$$

Integrating both sides

$$\frac{x^3}{3} = \frac{\pi^2}{3}x + 4\sum(-1)^n \frac{\sin nx}{n^3} + C_1$$

at $x=0$, $C_1 = 0$

Integrating again; $\frac{x^4}{12} = \frac{\pi^2 x^2}{6} - 4\sum(-1)^n \frac{\cos nx}{n^4} + C_2$

at $x=0$, $C_2 = 4\sum(-1)^n \frac{1}{n^4}$ at $x=\pi$, $\frac{\pi^4}{12} = \frac{\pi^4}{6} - 4\sum(-1)^n \frac{\cos n\pi}{n^4} + \ldots$

$$\Rightarrow -\frac{\pi^4}{12} = -4\sum\frac{1}{n^4} + 4\sum\frac{1}{n^4}\ldots$$

or $$\frac{\pi^4}{12} = 4\sum_{n=\text{odd}}\frac{1}{n^4}(1-(-1)^n)$$

$$\frac{\pi^4}{12} = 8\sum_{n=\text{odd}}\frac{1}{n^4}$$

or, $$\sum\frac{1}{n^4} = \frac{\pi^4}{96}$$

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Thanksss