Topic: Fourier series examples
Course: B.Sc/ Physics

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$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin kx\,dx$

$= \frac{1}{\pi}\left\{-\left[\frac{x}{k}\cos kx\right]_0^{\pi} + \frac{1}{k^2}\left[\sin kx\right]_0^{\pi}\right\}$

$= -\frac{1}{\pi}\left\{\frac{\pi}{k}\cos k\pi + 0\right\} = -\frac{\cos k\pi}{k}$.

$\cos k\pi = (-1)^k$

$\therefore a_0 = \frac{\pi}{2}$, $a_k = \frac{1}{\pi k^2}(\cos k\pi - 1)$, $b_k = -\frac{\cos k\pi}{k}$.

If k is even then $\cos k\pi = 1 \therefore a_k = 0$, $b_k = -\frac{1}{k}$

If k is odd " $\cos k\pi = -1 \therefore a_k = \frac{-2}{\pi k^2}$, $b_k = \frac{1}{k}$

Hence $b_k = (-1)^k\left(-\frac{1}{k}\right)$.

Since $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos kx + \sum_{k=1}^{\infty} b_k \sin kx$

$= \frac{\pi}{4} + a_1\cos x + a_2\cos 2x + \dots + b_1\sin x + b_2\sin 2x + \dots$

$= \frac{\pi}{4} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\cos k\pi - 1}{k^2}\cos kx - \sum_{k=1}^{\infty}(-1)^k\frac{\sin kx}{k}$.

Now $\sum_{n=1}^{\infty}(\cos n\pi - 1)\frac{\cos nx}{n^2} = -\frac{2\cos x}{1} - \frac{2\cos 3x}{9} - \frac{2\cos 5x}{25}$

$= -2\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$

Thus $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum\frac{\cos(2n-1)x}{(2n-1)^2} - \sum(-1)^n\frac{\sin nx}{n}$.

Putting $x = 0$, we get

$f(0) = 0 = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$

or, $1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$, (proved).

Q Obtain the Fourier series expansion of following function with period 2π.

$$f(x) = \begin{cases} x(\pi + x) & \text{for } -\pi \le x \le 0 \\ x(\pi - x) & \text{for } 0 \le x \le \pi. \end{cases}$$

for prove that $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots = \frac{\pi^3}{32}$

Sol $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} x(\pi+x)\,dx + \int_{0}^{\pi} x(\pi-x)\,dx\right.$

$$= \frac{1}{\pi}\left\{\left|\pi\frac{x^2}{2} + \frac{x^3}{3}\right|_{-\pi}^{0} + \left|\pi\frac{x^2}{2} - \frac{x^3}{3}\right|_{0}^{\pi}\right\}.$$

$$= \frac{1}{\pi}\left\{\frac{\pi^3}{2} - \frac{\pi^3}{3}\right\} + \frac{1}{\pi}\left\{\frac{\pi^3}{2} - \frac{\pi^3}{3}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{2\pi^3}{2} - \frac{2\pi^3}{3}\right. = \frac{2\pi^3}{\pi}\left(\frac{1}{2} - \frac{1}{3}\right) = 2\pi^2 \times \frac{1}{6} = \frac{\pi^2}{3}.$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}(\pi x + x^2)\cos kx\,dx + \int_{0}^{\pi}(\pi x - x^2)\,\cancel{\sin}\cos kx\,dx\right.$$

$$= \frac{1}{\pi}\left\{\pi\int_{-\pi}^{0} x\cos kx + \int_{-\pi}^{0} x^2\cos kx + \pi\int_{0}^{\pi} x\,\cancel{\sin}\cos kx - \int_{0}^{\pi} x^2\,\cancel{\sin}\cos kx\right\}$$

$$\cancel{= \frac{1}{\pi}\left\{\pi\left[\left(\frac{x}{k}\sin kx\right)_{-\pi}^{0} + \frac{1}{k^2}(\cos kx)_{-\pi}^{0} + \pi\right.\right.}$$

$$= \frac{1}{\pi}\left\{\pi\int_{\pi}^{0}(-z)\cos k(-z)(-dz) + \int_{\pi}^{0}(-z)^2\cos k(-z)(-dz) + \pi\int_{0}^{\pi} x\cos kx - \int_{0}^{\pi} x^2\cos\right.$$

$$= \int_{\pi}^{0} z\cos kz\,dz + \frac{1}{\pi}\int_{\pi}^{0} z^2\cos kz\,dz + \pi\int_{0}^{\pi} x\cos kx - \frac{1}{\pi}\int_{0}^{\pi} x^2\cos kx$$

$$= -\int_{0}^{\pi} x\cos kx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x^2\cos kx\,dx + \int_{0}^{\pi} x\cos kx - \frac{1}{\pi}\int_{0}^{\pi} x^2\cos kx$$

$$= 0.$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x(\pi+x)\sin kx\,dx + \int_{0}^{x} x(\pi-x)\sin kx\,dx\right.$$

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