Topic: Fourier series examples
Course: B.Sc/ Physics

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Q. The function $f(x) = x^2$ is defined within the interval $-\pi \le x \le \pi$, and outside it is periodic.

(i) Show whether the function is even or odd within $-\pi \le x \le \pi$.

(ii) Expand $f(x)$ in a F-series to show

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx.$$

(iii) Use the above result to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

A-(i) $f(-x) = (-x)^2 = x^2 = f(x)$. Hence the function is even. Sine term will be absent. $\omega = \frac{2\pi}{x_2 - x_1} = \frac{2\pi}{2\pi} = 1$

(ii) The F-series for $f(x)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{3\pi}[x^3]_{-\pi}^{\pi}$$

$$= \frac{1}{3\pi}(\pi^3 + \pi^3) = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}.$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos kx\,dx$$

$$= \frac{1}{\pi}\left[\underbrace{\left(x^2 \sin kx\right)_{-\pi}^{\pi}}_{=0} - \int_{-\pi}^{\pi} \frac{2x}{k}\sin kx\,dx\right]$$

$$= -\frac{2}{k\pi}\int_{-\pi}^{\pi} x\sin kx\,dx = -\frac{2}{k\pi}\left[-\left(\frac{x}{k}\cos kx\right)_{-\pi}^{\pi} + \frac{1}{k}\int_{-\pi}^{\pi}\cos kx\,dx\right]$$

$$= -\frac{2}{k\pi}\left\{-\left(\frac{\pi}{k}\cos k\pi + \frac{\pi}{k}\cos k\pi\right) + \frac{1}{k^2}\left|\sin kx\right|_{-\pi}^{\pi}\right\}$$

$$= -\frac{2}{k\pi}\left[-\frac{2\pi}{k}\cos k\pi\right] = +\frac{4\pi}{k^2}\cos k\pi.$$

$$= \frac{4}{k^2}(-1)^k$$

$$f(x) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2}(-1)^k \cos kx.$$

(ii) putting $x=\pi$ in the above $(-1)^n \cos n\pi = (-1)^{2n} = 1$

$$f(\pi) = \pi^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\therefore \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{4}\cdot\frac{2\pi^2}{3} = \frac{\pi^2}{6} \quad \text{Proved}$$

Q/ Expand $f(x) = A$, where A is a constant, in the range $0 < x < \pi$, in a Fourier series. Hence show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Sol: we expand $f(x)$ in half-range sine series in the range $0 < x < \pi$. The Fourier series is

$$F(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2A}{\pi}\int_0^{\pi} \sin nx\, dx$

$$= \frac{2A}{\pi n}\left(-\cos nx\right)_0^{\pi} = \frac{2A}{\pi n}(1 - \cos n\pi).$$

If n is odd, $\cos n\pi = -1$, $b_n = \frac{4A}{\pi n}$,

" n is even, $\cos n\pi = 1$ & $b_n = 0$.

Hence $f(x) = A = \frac{4A}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\sin nx = \frac{4A}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \cdots\right]$

Q/ Show that the function

$$f(x) = \begin{cases} 0 & \text{for } -\pi \le x < 0 \\ x & \text{for } 0 \le x < \pi \end{cases}$$

Sol. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right\}$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\,dx\right\} = \frac{\pi}{2}$$

$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos kx\,dx$

$$= \frac{1}{\pi}\left\{\left[\frac{x}{k}\sin kx\right]_0^{\pi} + \frac{1}{k^2}\left[\cos kx\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{k}\sin k\pi + 0\right\} + \frac{1}{\pi k^2}\left[\cos k\pi - \cos 0\right]$$

$$= \frac{1}{\pi k^2}\left[\cos k\pi - 1\right]$$

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Thanksss